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Full Length Research Paper

Numerical solution of the Boussinesq equation: Application to the agricultural drainage

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Water flow to subterranean drains is described by the Boussinesq equation. To solve this equation, analytical solutions comprising constants, such as the transmissivity and drainable porosity have been developed; however, these solutions assume that free surface of the water falls instantly over the drains. The aim of this investigation is to present a finite difference solution of the differential equation using a drainable porosity variable and a fractal radiation condition. Here, two schemes are presented: the first one, with an explicit head and drainable porosity, both joined by a functional relationship, called mixed formulation; and the second one, called head formulation, with only the head. By using a lineal analytical solution, both methods have been validated and the nonlinear part was stable and brief. The proposed numerical solution is useful for the hydraulic characterization of soils with inverse modelation and to improve the designs of agricultural drainage systems, when taking into consideration that the assumptions of the classical solution have been eliminated. To evaluate the descriptive capacity of the numerical solution, these solutions were used to describe a drainage experiment performed in the laboratory. The results show that the cumulative drained depth is well represented by these solutions with the fractal radiation and the variable drainable porosity.

Key words: Fractal radiation condition, variable drainable porosity, analytical solution, mixed formulation, head formulation.

INTRODUCTION

Subsurface drainage systems are used to control the depth of the water-table and to reduce the water of the root zone or prevent soil salinity in the soil profile. The analysis of water dynamics in these systems has been studied accepting the validity of Darcy's law (1956) and depending on the scale of study; two differential equations can be used. The first is the Richards equation (1931), resulting from the application of the mass conservation principle in the flow of water in an elemental volume of porous medium, and the Darcy's law which permits the consideration of the geometry of the drains in boundary conditions, but the simulation of water dynamics with two or three dimensional numerical solutions can be arduous (Zavala et al., 2007). The second equation is the Boussinesq equation (1904) for an unconfined aquifer, resulting from the application of the mass conservation principle, Darcy's law, and the Dupuit-Forcheimer hypothesis concerning the hydro-statics distribution of pressure (Bear, 1972), weighted soil properties and the vertical system. It is at most a two-dimensional equation. The aquifer is modeled on the ground and the geometry of the drains is introduced as mathematical lines or dots in a two-dimensional or one-dimensional analysis, respectively.

The one-dimensional Boussinesq equation has been the basis for developing approximate analytical solutions of the water dynamics in a drainage system within either a permanent or transitory regimen (Hooghoudt, 1940;

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Dumm, 1954; Shukla et al., 1999; Upadhyaya and Chauhan, 2001; van de Giesen et al., 2005; Spanoudaki et al., 2010). In the derivation of the Glover-Dumm equation for the transitory regimen, it is assumed that the aquifer transmissivity and drainable porosity are constant and the free surface falls instantly over the drains. Due to these three assumptions it does not adequately represent actual conditions. The solution is likely to be of limited applicability. However, considering the most representative conditions lead to analytical difficulties, the use of numerical methods is required to develop solutions of the Boussinesq equation (Hall and Moench, 1972; Hogarth et al., 1999). In the research done by Zavala et al. (2007) the real boundary conditions are analyzed in detail. The authors, basing their proposals on the fractal geometry concepts and in drainage experience, recommended a fractal radiation condition, which contains the lineal radiation (Fuentes et al., 1997).

With respect to drainable porosity, Fuentes et al. (2009) based in the drainable depth and drained depth concepts, and in drainage experience, propose an analytical expression which involves the soil water retention curve. These authors used the finite element method to solve a one-dimensional Boussinesq equation with good results in terms of stability, convergence and accuracy of the solution. In a one-dimensional scheme the finite element method can be equivalent to the finite difference method (Russell and Wheeler, 1983). A finite difference scheme, based in the Laasonen scheme, was develop by Zataráin et al. (1998) in order to numerically solve the Richards equation, applied to water infiltration phenomenon in the soil with excellent results. In addition to its high accuracy, stability and convergence, the scheme has the added advantage of its intuitive nature, based on a local mass balance. This scheme can also be used to solve the agricultural drainage one-dimensional Boussinesq equation. The aim of this investigation is to show a numerical solution of the Boussinesq equation, with a different method based on a local mass balance. The drainable porosity is considered variable and the boundary conditions at the drains are fractal radiation.

THEORY

The Boussinesq equation

In the study of the water dynamics in subsurface agricultural drainage systems by using the Boussinesq equation, the variations in hydraulic head along the drain pipes (direction y) are negligible with respect to head variations in the cross section (direction x). It is the one-dimensional Boussinesq equation which is a result of the continuity equation and the Darcy's law, namely:

$$\mu(H)\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[T(H)\frac{\partial H}{\partial x}\right] + R_w \tag{1}$$

where $\mu(H)$ is the storage capacity, $H = H(x,t)$ is the elevations of the free surface or hydraulic head above the impervious layer, and is a function the horizontal coordinate (x) and the time (t), $T(H)$ is the transmissivity given by $T(H) = K_s H$, R_w is the recharge volume in the unit time by area unit aquifer, and K_s is the saturated hydraulic conductivity. The storage capacity is defined by:

$$\mu(H) = \frac{dW}{dH} = \upsilon(H) + H\frac{d\upsilon}{dH} \tag{2}$$

where $\upsilon(H)$ is the drainable porosity as a function of the head, $W = \upsilon H$ is the drainable depth. The equality of $\mu = \upsilon$ is when the drainable porosity is independent of the head (Fuentes et al., 2009).

The drainable porosity

An expression of the storage capacity is given by Fuentes et al. (2009):

$$\mu(H) = \theta_s - \theta(H - H_s) \tag{3}$$

where θ_s is the saturated volumetric water content, and $\theta(H - H_s)$ represents the water content evolution in the position $z = H_s$, while the free surface decreases, and z is the elevation of ground surface.

The drainable porosity is deduced from joining Equations (2) and (3), namely (Fuentes et al., 2009):

$$\upsilon(H) = \frac{1}{H}\int_0^H \mu(\bar{H})\,d\bar{H} = \frac{1}{H}\int_0^H \left[\theta_s - \theta(\bar{H} - H_s)\right]d\bar{H} \tag{4}$$

where $\bar{H}$ is the integration variable.

To calculate the storage capacity and the drainable porosity it is necessary to provide the soil water retention curve. The model of van Genuchten (1980) has been widely accepted in field and laboratory studies, namely:

$$\theta(\psi) = \theta_r + (\theta_s - \theta_r)\left[1 + \left(\frac{\psi}{\psi_d}\right)^n\right]^{-m} \tag{5}$$

where θ_r is the residual volumetric water content, ψ_d is the pressure scale parameter, and m and n are positives form parameters.

The introduction of Equation (5) in Equations (3) and (4) permitted to obtain the following storage capacity and drainable porosity:

$$\mu(H)=(\theta_s-\theta_r)\left\{1-\left[1+\left(\frac{H_s-H}{|\psi_d|}\right)^n\right]^{-m}\right\} \tag{6}$$

$$\upsilon(H)=(\theta_s-\theta_r)\left[1-\frac{|\psi_d|}{H}\int_{(H_s-H)/|\psi_d|}^{H_s/|\psi_d|}\left(1+\psi_*^n\right)^{-m}d\psi_*\right] \tag{7}$$

The drainable porosity does not have a closed analytical form and can be calculated by numerical integration. A closed form can be obtained from the Fujita-Parlange model; this is obtained from the Fujita (1952) diffusivity equation and from the relationship between hydraulic conductivity and hydraulic diffusivity of Parlange et al. (1982), (Fuentes et al., 1992):

$$D(\Theta)=\left(\frac{K_s\lambda_c}{\theta_s-\theta_r}\right)\frac{1-\alpha}{(1-\alpha\Theta)^2} \tag{8}$$

$$K(\Theta)=K_s\frac{\Theta\left[1-\beta+(\beta-\alpha)\Theta\right]}{1-\alpha\Theta} \tag{9}$$

where $\Theta=(\theta-\theta_r)/(\theta_s-\theta_r)$ is the effective saturation, α and β are non dimensional form parameters that $0<\alpha<1$ and $0<\beta<1$; and λ_c is the Bouwer scale (1964).. The retention curve is obtained from the hydraulic diffusivity $D(\theta)=K(\theta)d\psi/d\theta$, considering $\theta=\theta_s$ when $\psi=0$, is obtained:

$$\psi(\Theta)=\psi_c\left\{\frac{\alpha}{\beta}\ln\left[\frac{1-\alpha\Theta}{(1-\alpha)\Theta}\right]+\frac{\beta-\alpha}{\beta(1-\beta)}\ln\left[\frac{1-\beta+(\beta-\alpha)\Theta}{(1-\alpha)\Theta}\right]\right\} \tag{10}$$

where $\psi_c=-\lambda_c$.

The drainable porosity is obtained from Equations (4) and (10):

$$\upsilon(H)=(\theta_s-\theta_r)\left\{1-\frac{\lambda_c}{\beta H}\ln\left[\frac{1-\beta+(\beta-\alpha)\Theta}{1-\alpha\Theta}\right]\Bigg|_{\Theta(-H_s)}^{\Theta(H-H_s)}\right\} \tag{11}$$

The $\theta(\psi)$ function is implicit in Equation (10) and therefore the $\upsilon(H)$ function. If $\alpha=\beta$ is accepted, this then leads to the hydraulic conductivity model proposed by Gardner (1958): $K(\psi)=K_s\exp(\psi/\lambda_c)$, used in theoretical studies. The corresponding $\theta(\psi)$ curve is:

$$\theta(\psi)=\theta_r+\frac{\theta_s-\theta_r}{\alpha+(1-\alpha)\exp(\psi/\psi_c)} \tag{12}$$

The storage capacity is obtained from Equation (3):

$$\mu(H)=(\theta_s-\theta_r)\left\{1-\left[\alpha+(1-\alpha)\exp\left(\frac{H_s-H}{\lambda_c}\right)\right]^{-1}\right\} \tag{13}$$

and the drainable porosity is obtained from Equation (11):

$$\upsilon(H)=(\theta_s-\theta_r)\left\{1-\frac{\lambda_c}{\alpha H}\ln\left[\frac{1-\alpha+\alpha\exp\left[(H-H_s)/\lambda_c\right]}{1-\alpha+\alpha\exp\left[-H_s/\lambda_c\right]}\right]\right\} \tag{14}$$

The drainable depth is (Fuentes et al., 2009):

$$\ell(H)=(\theta_s-\theta_r)\left\{(H_s-H)-\left(\frac{\lambda_c}{\alpha}\right)\ln\left[\frac{1}{1-\alpha+\alpha\exp\left(-(H_s-H)/\lambda_c\right)}\right]\right\} \tag{15}$$

The saturated volumetric water content can be assimilated to the soil porosity (ϕ), this is calculated with the formula $\phi=1-\rho_t/\rho_o$, where (ρ_t) is the bulk density and (ρ_o) is the particles density; the residual volumetric water content (θ_r) is considered to be zero.

Initial and boundary conditions

The hydraulic head counting from above the impermeable barrier $H(x,t)$ is associated with the head $h(x,t)$ counting from above the drains by using:

$$H(x,t)=D_o+h(x,t) \tag{16}$$

where D_o is the distance from the impermeable barrier.

Transversal variation of h at the beginning is considered as the initial condition:

$$h(x,0)=h_s(x) \tag{17}$$

Regarding the boundary conditions in $x=0$ and $x=L$, many forms have been assumed. The Glover-Dumm solution is established assuming that the head on the drain instantly reaches zero value (Dumm, 1954) this is the Dirichlet condition or first order. The Fuentes et al. (1997) solution is obtained from experimental analysis (Fragoza et al., 2003), supported by the argument that the Darcy flow in the drains is proportional to the head $(q \propto h)$; this is radiation lineal condition or third order. The lineal radiation flux can be expressed as $q=-\kappa K_s h/L$, where κ is the non dimensional conductance coefficient of the soil-drain interface. From the Fuentes et al. (1997) solution the Glover-Dumm solution can be obtained if this coefficient is infinite. In this line of research, Zavala et al. (2007) proposed a power law between the flux and head, $q=q_s(h/h_s)^{2s}$, where h_s is the value on the drain in the initial time, q_s is the corresponding flux to h_s and it is function of the soil-drain interface characteristic. For the s parameter, the authors argued that it is defined by $s=D/E$, where D is the effective fractal dimension to the soil-drain interface, and $E=3$ is the Euclidean dimension of physical space. The s and effective porosity relation of interface is obtained from the equation given by Fuentes et al. (2001):

$$(1-\phi)^s+\phi^{2s}=1 \tag{18}$$

Thus, the fractal radiation condition for the Boussinesq equation is obtained as follows:

$$-K_s\frac{\partial h}{\partial x}\pm q_s\left(\frac{h}{h_s}\right)^{2s}=0,\ x=0,L \tag{19}$$

where the positive sign corresponds to $x=0$ and the negative sign to $x=L$. L is the distance between drains. Equation (19) contains as particular cases the lineal radiation condition when $s=1/2$ and the quadratic radiation condition when $s=1$. In a system of parallel drains, the drained water flows by length unit at each drain is:

$$Q_d(t)=2\left[D_o+h(0,t)\right]q_s\left[h(0,t)/h_s\right]^{2s}, \tag{20}$$

and the cumulative drained depth by:

$$\ell(t)=\frac{1}{L}\int_0^t Q_d(\bar{t})\,d\bar{t} \tag{21}$$

where $\bar{t}$ is the integration variable.

NUMERICAL SOLUTION

Numerical schemes

The one-dimensional Boussinesq equation is solved using the difference finite method, adapting the numerical scheme proposed by Zataráin et al. (1998) for a similar problem, but at the scale of the Richards equation. The scale adjustment of the Boussinesq equation requires the discretization of the domains as shown in Figure 1. To solve the numerical solution of Equation, interpolation parameters are introduced:

$$\gamma=\frac{x_{i+\gamma}-x_i}{x_{i+1}-x_i},\ \omega=\frac{t_{j+\omega}-t_j}{t_{j+1}-t_j} \tag{22}$$

where $0\le\gamma\le1$ and $0\le\omega\le1$; $i=1,2,...$ and $j=1,2,...$ are the space and time indices, respectively. The dependent variable (H) in an intermediate node $i+\gamma$ for all j is estimated as:

$$H_{i+\gamma}^j=(1-\gamma)H_i^j+\gamma H_{i+1}^j \tag{23}$$

while the intermediate time $j+\omega$ for all i is estimated as:

$$H_i^{j+\omega}=(1-\omega)H_i^j+\omega H_i^{j+1} \tag{24}$$

The continuity equation applies in time $t_{j+\omega}$, the time derivative can be discretized according to the following two methods:

$$\left.\frac{\partial(\upsilon H)}{\partial t}\right|_i^{j+\omega}=\frac{\upsilon_i^{j+1}H_i^{j+1}-\upsilon_i^jH_i^j}{\Delta t_j},\ \Delta t_j=t_{j+1}-t_j \tag{25}$$

$$\left.\frac{\partial(\upsilon H)}{\partial t}\right|_i^{j+\omega}=\mu_i^{j+\omega}\frac{H_i^{j+1}-H_i^j}{\Delta t_j} \tag{26}$$

The first one is called a mixed scheme, and the second

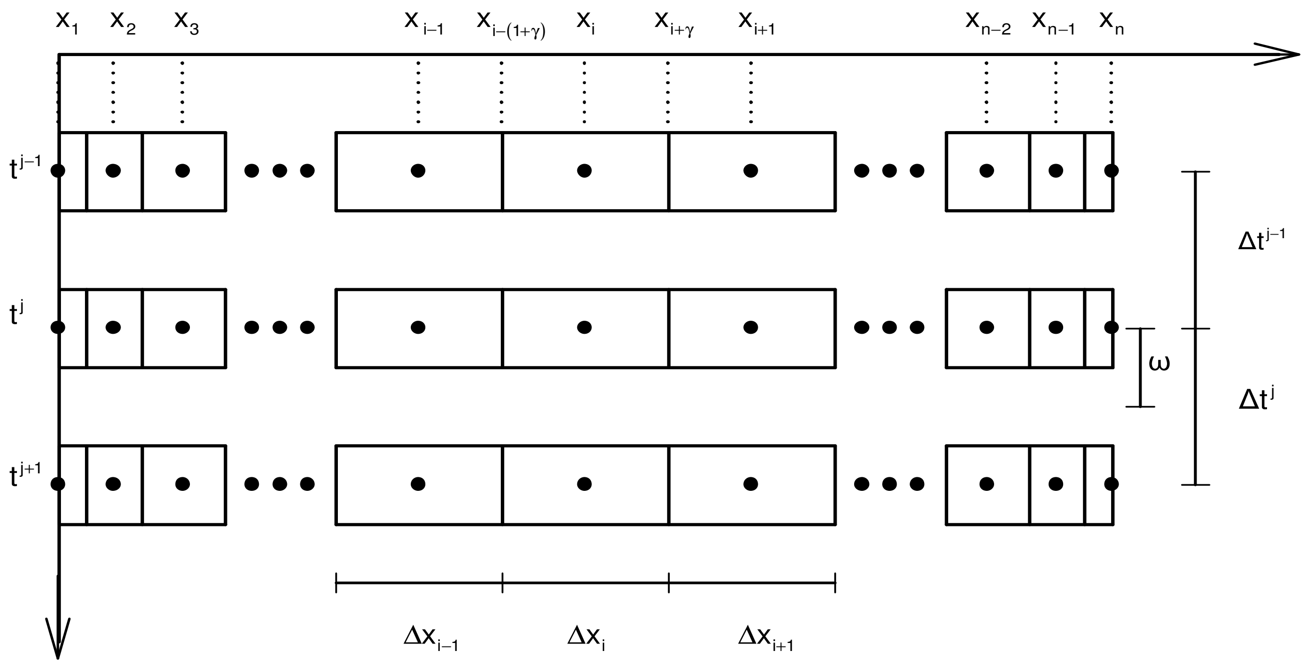


Figure 1. Scheme of the solution domain of Boussinesq equation: where t is the time (temporal coordinate), x is the length (spatial coordinate), ω is the time interpolation factor, γ is the space interpolation factor, i is the spatial index, j is the temporal index, Δz is the spatial increment and Δt is the temporal increment.

one a head scheme. In the first, the head and the water volume appears explicitly, while in the second only the head. Both formulations are the same when the drainable porosity is independent of the head and the head scheme does not require numerical integration in Equation to calculate the drainable porosity. The spatial derivate discretization around the node i-th is:

$$\left.\frac{\partial (Hq)}{\partial x}\right|_i^{j+\omega} = \frac{(Hq)_{i+\gamma}^{j+\omega} - (Hq)_{i-(1-\gamma)}^{j+\omega}}{\Delta x_i},$$

$$\Delta x_i = (1-\gamma)(x_i - x_{i-1}) + \gamma(x_{i+1} - x_i) \quad (27)$$

The water flow by length unit defined in the intermediate nodes is:

$$\left.(Hq)\right|_{i+\gamma}^{j+\omega} = -T_{i+\gamma}^{j+\omega}\frac{H_{i+1}^{j+\omega} - H_i^{j+\omega}}{x_{i+1} - x_i},\; T_{i+\gamma}^{j+\omega} = T\left(H_{i+\gamma}^{j+\omega}\right) \quad (28)$$

$$\left.(Hq)\right|_{i-(1-\gamma)}^{j+\omega} = -T_{i-(1-\gamma)}^{j+\omega}\frac{H_i^{j+\omega} - H_{i-1}^{j+\omega}}{x_i - x_{i-1}},\; T_{i-(1-\gamma)}^{j+\omega} = T\left(H_{i-(1-\gamma)}^{j+\omega}\right) \quad (29)$$

The heads on different nodes and intermediate times are obtained from Equation (24); these are introduced in Equations (28), (29), and these in Equation (27). The Equations (25) and (27) are put into the continuity equation and similar terms are associated, resulting in the following algebraic equations system:

$$A_i H_{i-1}^{j+1} + B_i H_i^{j+1} + D_i H_{i+1}^{j+1} = E_i,\; i = 2, 3, ..., n-1 \quad (30)$$

Where $A_i = -\dfrac{\omega T_{i-(1-\gamma)}^{j+\omega}}{\Delta x_i (x_i - x_{i-1})}$ (31)

$$B_i = \frac{\omega}{\Delta x_i}\left(\frac{T_{i+\gamma}^{j+\omega}}{x_{i+1} - x_i} + \frac{T_{i-(1-\gamma)}^{j+\omega}}{x_i - x_{i-1}}\right) + \frac{\upsilon_i^{j+1}}{\Delta t_j} \quad (32)$$

$$D_i = -\frac{\omega T_{i+\gamma}^{j+\omega}}{\Delta x_i (x_{i+1} - x_i)} \quad (33)$$

$$E_i = R_{wi}^{\;j+\omega} + \frac{(1-\omega)}{\Delta x_i}\left(\frac{T_{i-(1-\gamma)}^{j+\omega} H_{i-1}^j}{x_i - x_{i-1}} + \frac{T_{i+\gamma}^{j+\omega} H_{i+1}^j}{x_{i+1} - x_i}\right) + \left[\frac{\upsilon_i^j}{\Delta t_j} - \frac{(1-\omega)}{\Delta x_i}\left(\frac{T_{i+\gamma}^{j+\omega}}{x_{i+1} - x_i} + \frac{T_{i-(1-\gamma)}^{j+\omega}}{x_i - x_{i-1}}\right)\right] H_i^j \quad (34)$$

For the head scheme, Equation (26), the coefficients B_i and E_i should be redefined, replacing υ_i^{j+1} and υ_i^j in Equations (32) and (34) by $\mu_i^{j+\omega}$. The system (30) forms a tridiagonal matrix and can be solved using the Thomas algorithm. Due to the fact that the coefficients in the system (30) depend on the pressure H_i^{j+1}, these are updated and the algorithm is applied again until the following estimator from the solution in t_{j+1} is equal, given an error criterion, to the previous estimator.

Radiation condition boundary

For linearizing the boundary condition, one generalization of the conductance coefficient, presented in the discussions concerning Equation (19) is introduced as:

$$\kappa = \frac{q_s}{K_s}\frac{L}{h_s}\left(\frac{h}{h_s}\right)^{2s-1} \tag{35}$$

It should be noted that κ depends on the solution itself; however, as the process of solutions of system (30) is iterative, this parameter is calculated based on the previous estimation.

Selection of the space (Δx) and time (Δt) increments

According to Zataráin et al. (1998) the domain discretization is done so that the increased $x_i - x_{i-1} = \delta x$ is constant for $i = 4, 5 ... N-2$ except in the drain vicinity. Thus, for $x_1 = 0$: i) $x_2 - x_1 = 0.4\delta x$, $x_3 - x_2 = 0.6\delta x$, $\Delta x_1 = 0.1\delta x$, $\Delta x_2 = 0.6\delta x$; y ii) $x_N = L$, $x_N - x_{N-1} = 0.4\delta x$, $\Delta x_{N-1} = 0.6\delta x$, $\Delta x_N = 0.1\delta x$. The interpolation value in the space is taken as $\gamma = 1/2$ in the domain, except in the first and last cells. The time discretization, given space's, follows the classical approach of writing the equations of motion in dimensionless form, valid in homogeneous media, to obtain the relation between the spatial and temporal characteristics.

Introducing dimensionless variables in the Boussinesq equation, Equation defined as: $x_* = x/L$, $t_* = t/\tau$, $H_* = H/H_s$, $\mu_* = \mu/\upsilon_s$, $R_{w*} = R_w L^2/T_s H_s$, where $\upsilon_s = \upsilon(H_s)$ and $T_s = K_s H_s$, it is possible to obtain the same Boussinesq equation with variables with asterisks if $\tau = \upsilon_s L^2/T_s$. Due to the parabolic nature of the differential equation, the parameter $M = (\Delta x_*)^2/\Delta t_*$ is defined, which can be found by comparing the finite difference solution with analytical solutions. The parameter value for the short times as recommended by Zataráin et al. (1998) is about $M \cong 0.1$.

Comparison with an analytical solution

In order to define the initial values of the interpolation parameters in space and time (γ and ω), the numerical solution is compared to one analytical solution obtained from the Boussinesq equation in a particular case. This solution has been developed by a linearization of the differential equation represented by a constant transmissivity, but with a linear radiation condition in the drains and includes the Glover-Dumm classical equation (Dumm, 1954). The values used for the simulation are the ones given by Fragoza et al. (2003): $L = 50$ m, $K_s = 0.557$ m/d, $\bar{\mu} = 0.1087\ m^3/m^3$, $\bar{T} = 2.5065\ m^2/d$, $D_o = 3.5$ m, $H_s = 5.0$ m and $\kappa = 1.5$. With the purpose of showing the effect of the parameter M in the numerical solution, many simulations were carried out, and the sum of squares error (SSE) obtained, assuming Δx as the constant and varying the values ω and M. The results are shown in Figure 2. In order to prevent the errors from remaining hidden in the short time, a calculation of the sum of squares errors was carried out for durations ranging from one day intervals up until 60 days. Notice that the SSE is lower with values of ω near to 1, and with values of M lower to 0.5. On the other hand, to decrease the value of ω, the errors in the solutions increase even with values lower than 0.5 in the parameter M, and the SSE increases significantly and conversely. Therefore, it was observed that when the value is M > 0.4 in the solution, instability is shown at the border, like that showing in Figure 3. In this case, the variation of decrease on free surface on the drain is shown with different interpolation steps (ω). The time discretization was done with $\Delta x = 1.00$ m and the time with $\Delta t = 0.01$ d, which corresponds to the value at $M \cong 4.33$; this value is higher than that recommended by Zataráin et al. (1998) for the short time. For the simulations done and shown in Figure 3, it was noted that the optimal interpolation step which permits the numerical solution to match the analytical solution, given an error criterion is $\omega = 0.98$. The same results are obtained when $\omega = 1.00$.

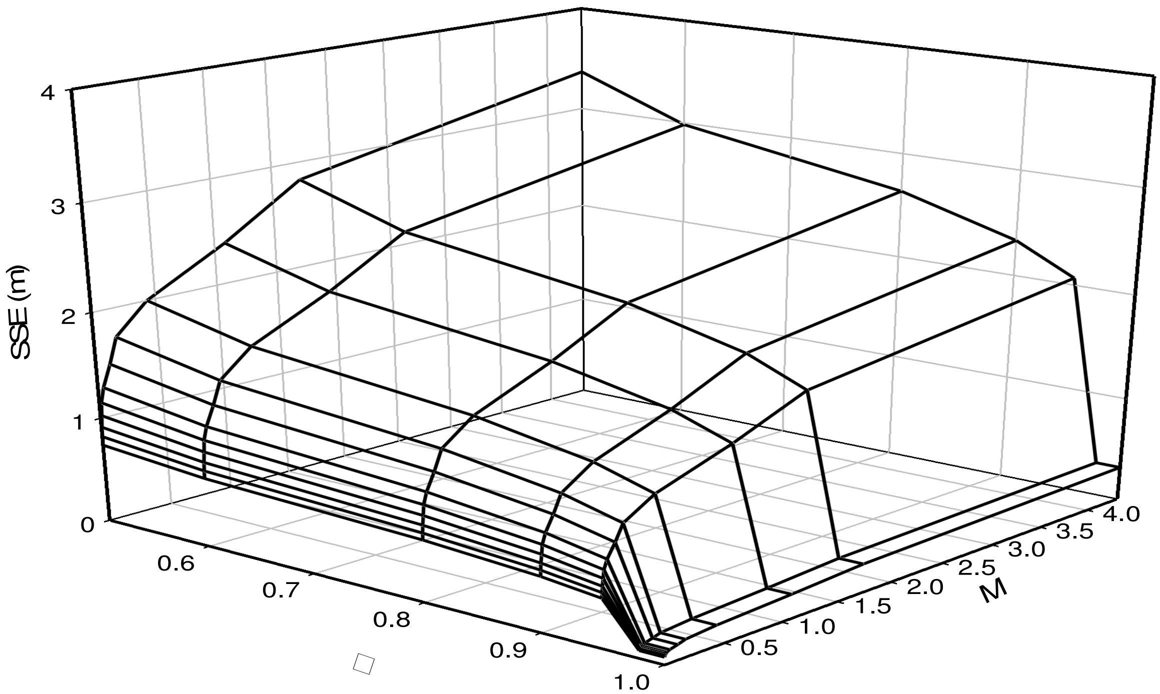


Figure 2. Sum of the square errors with different values of M and ω.

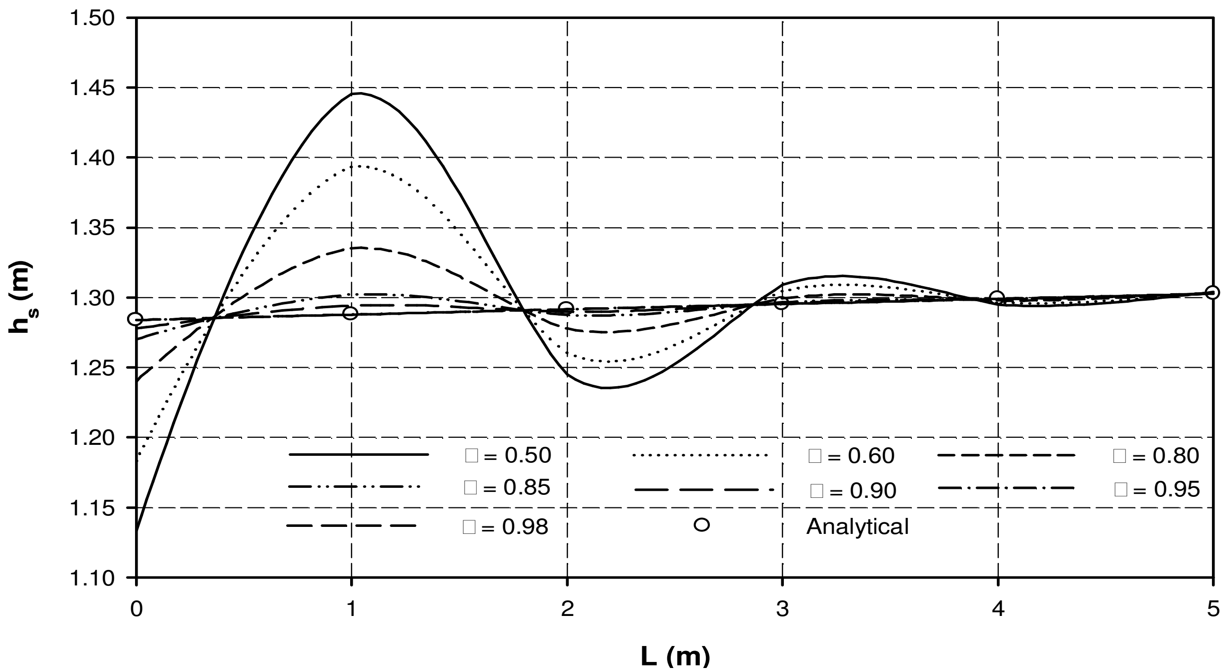


Figure 3. Evolution of the free surface on the drain with different values of time interpolation factor (ω).

Moreover, when the value of $M < 0.4$, the difference between the numerical solution and the analytical solution with different values of ω are minimal. An example of this is the one shown in Figure 4 where the profile decreases for two different values of M: $M \cong 0.43$ ($\Delta x = 1.00\ \mathrm{m}$ and $\Delta t = 0.10\ \mathrm{d}$) and $M \cong 0.04$ $\Delta x = 0.01\ \mathrm{m}$

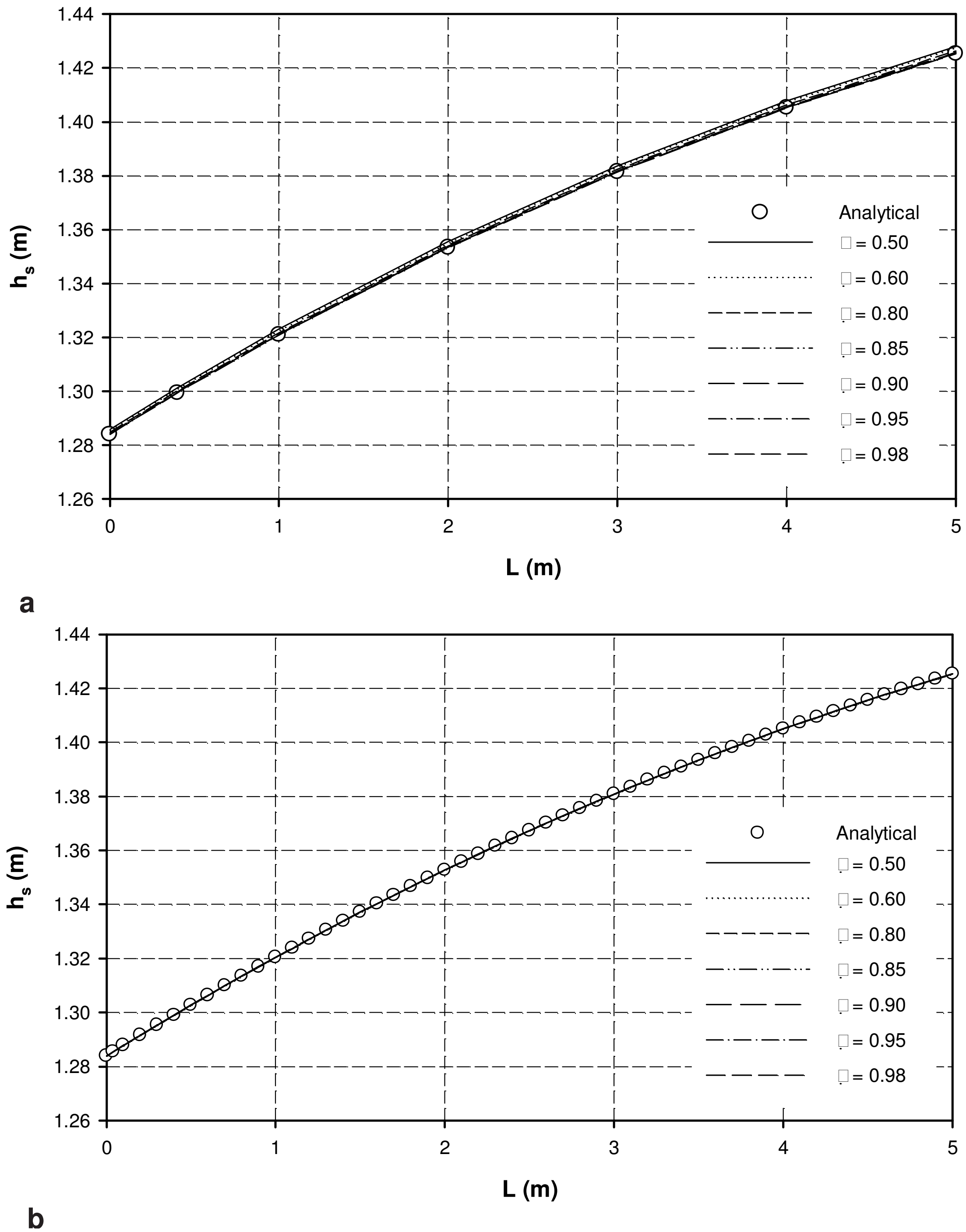


Figure 4. Evolution of the free surface, a) $M \cong 0.43$, b) $M \cong 0.04$.

(and $\Delta t = 0.0001\,d$). With the first value of M, it can be seen that the difference between the different values of ω are remarkable, and using the second value, the difference between the analytical solution and the proposed solution are minimal, given an error criterion. With the values $\Delta x = 0.01\,m$ and $\Delta t = 0.0001\,d$ previously obtained, a new simulation was carried out over a longer period of time; this simulation is shown in Figure 5. Here, the decrease of the free surface and the drained depth by unit soil area is shown. The results

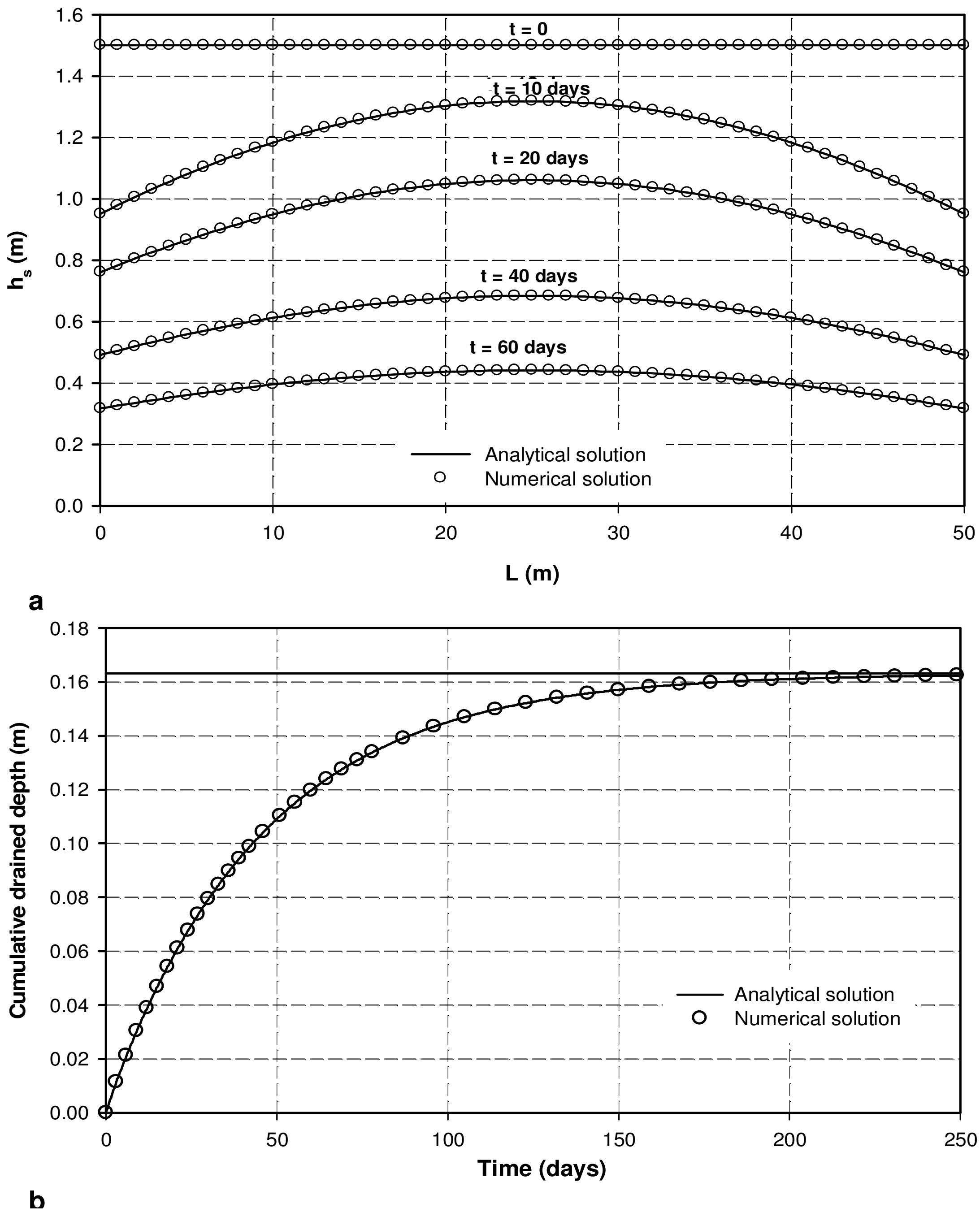


Figure 5. Comparison between the analytical and numerical solutions: a) Evolution of free surface, b) Evolution of the drained depth.

indicate that there is not a significant difference between the analytical solution and the finite difference solution.

Comparison between both numerical schemes

The mixed and head schemes are compared by accepting the values $\gamma = 0.5$ and $\omega = 0.98$. The soil used for this propose is the same as that used by Saucedo et al. (2003) with the values $\theta_s = 0.5245 \text{ cm}^3/\text{cm}^3$, $\theta_r = 0 \text{ cm}^3/\text{cm}^3$ and $K_s = 0.446 \text{ m/d}$. The values of the parameters of the hydrodynamics characteristics are: i) for Fujita and Parlange $\lambda_c = 0.521 \text{ m}$ and $\alpha = 0.98$; ii)

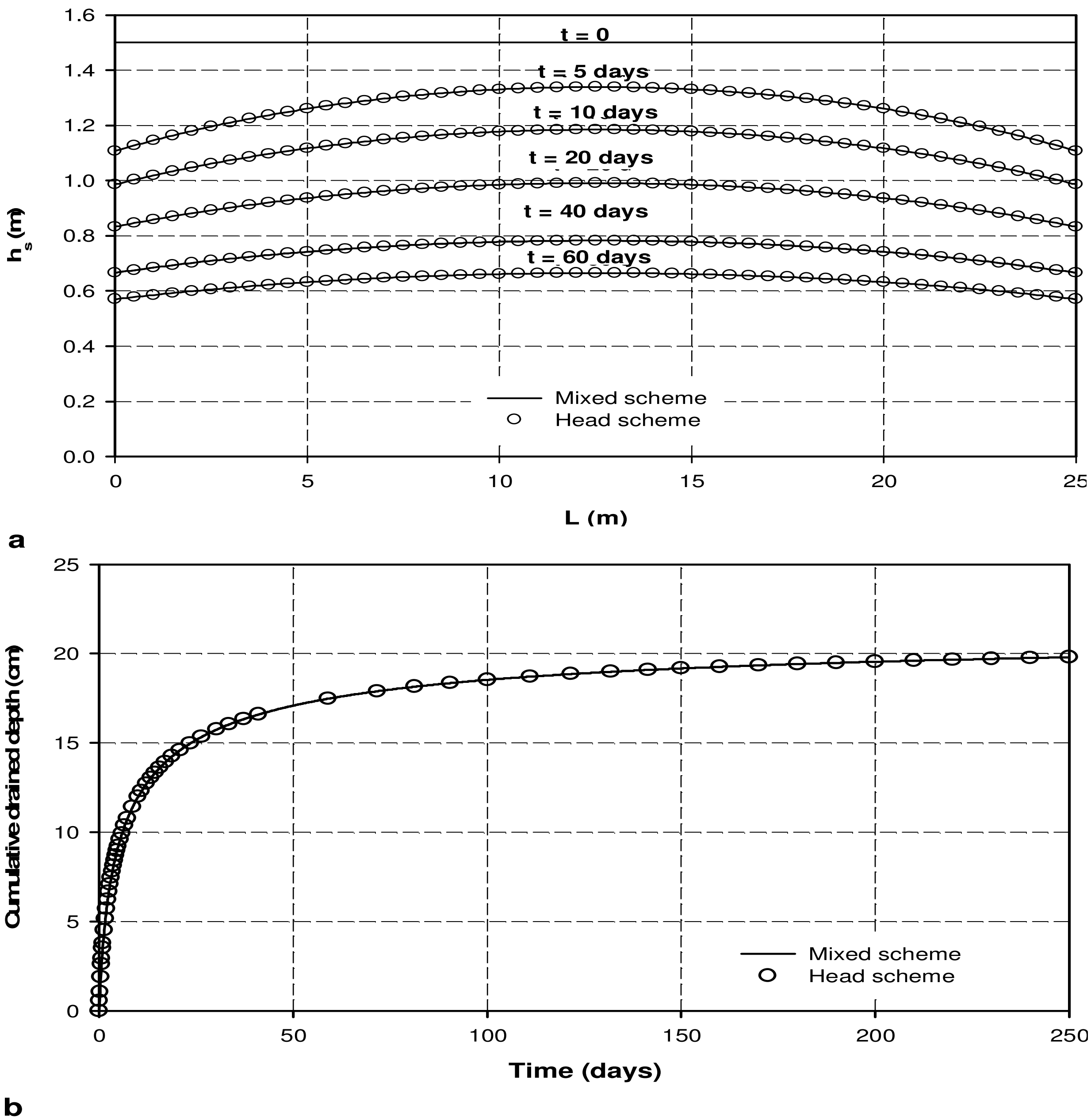
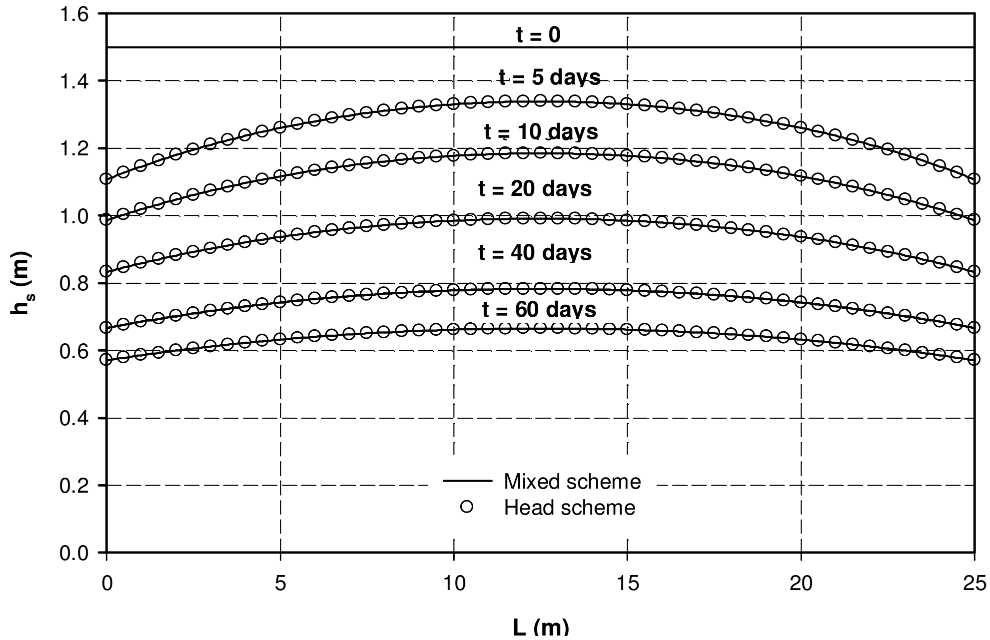


Figure 6. Comparison between both numerical schemes: a) Evolution of the free surface, b) Evolution of the drained depth, using the Fujita-Parlange hydrodynamics characteristics.

for van Genuchten with the Burdine restriction (1953) $m = 1 - 2/n$, $m = 0.066$ and $\psi_d = -0.15\ m$. To compare both schemes, one distance of drain separation is proposed $(L = 25\ m)$ and the drain position is $H_s = 1.5\ m$. The results of the numerical solutions using both hydrodynamics characteristics are shown in Figures 6 and 7. In Figure 6, the evolution of the free surface decreased and the drained water for time up to 60 and 250 days are shown, respectively, using the Fujita-Parlange hydrodynamics characteristics. Figure 7 shows the same results, but with the van Genuchten hydrodynamics characteristics. Notice that there are no differences between the mixed and head schemes, with each hydrodynamics characteristics.

These results demonstrate that using any of these schemes, the results will always be the same.



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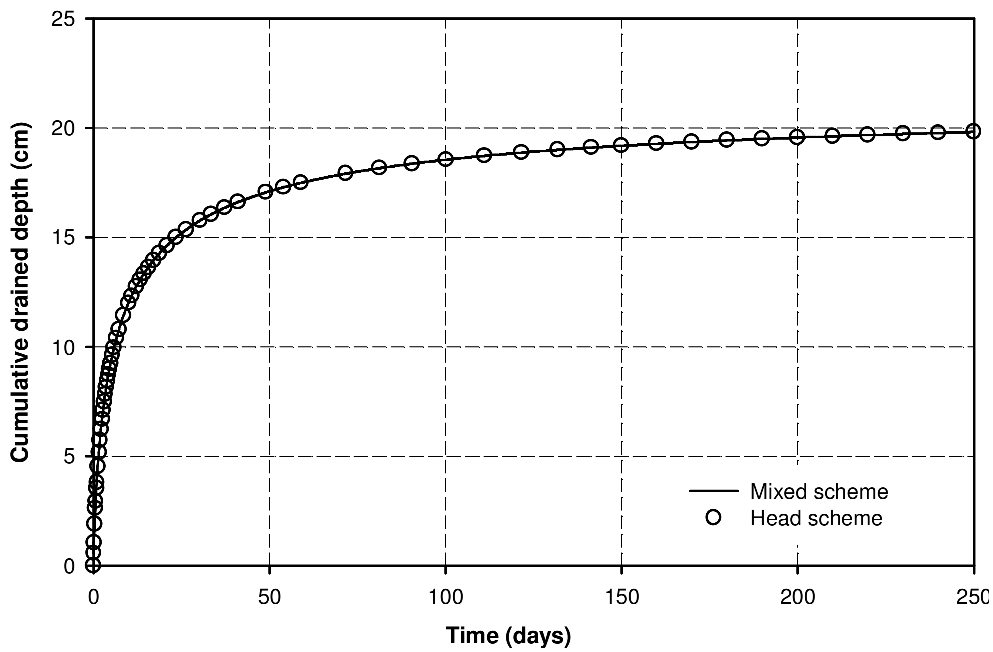

b

Figure 7. Comparison between both numerical schemes: a) Evolution of the free surface, b) Evolution of the drained depth, using the van Genuchten hydrodynamics characteristics.

APPLICATION

To evaluate the descriptive capacity of the numerical solution, a drainage experiment was conducted in a laboratory. The drainage module is the one used by Zavala et al. (2007). The module dimensions are: $L = 100\,\mathrm{cm}$, $H_s = 120\,\mathrm{cm}$ and $D_o = 25\,\mathrm{cm}$. The drain diameter is $d = 5\,\mathrm{cm}$ and the drain length is $\ell = 30\,\mathrm{cm}$. The module was filled with altered sample of silty soil of the Mexican region of Celaya, Guanajuato, passed through a 2 mm sieve; the soil was disposed on 5 cm thick layers, in order to maintain a constant apparent density. The soil was saturated by applying a constant water head on its surface until the entrapped air was virtually removed.

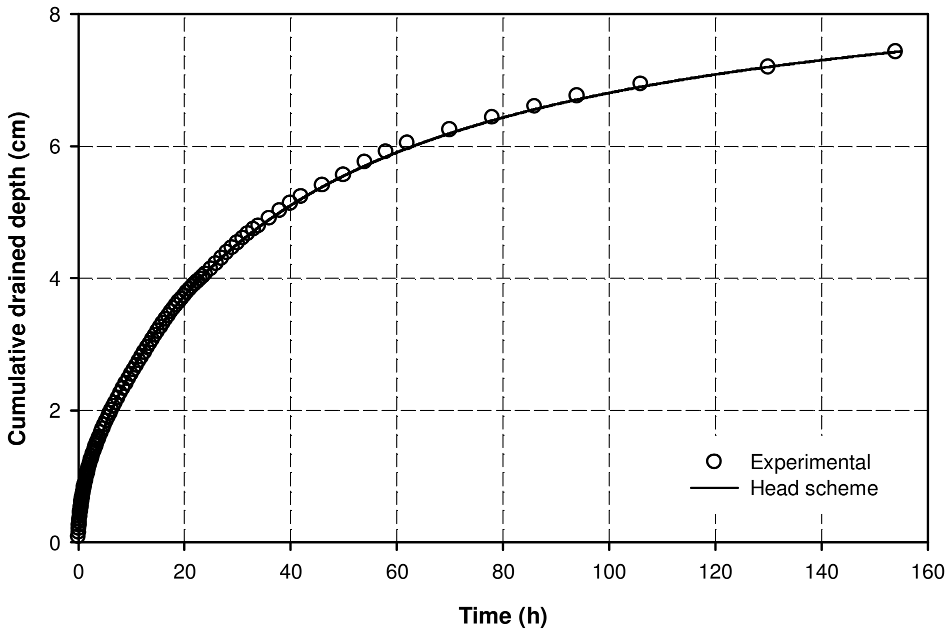

Figure 8. Evolution of the experimental drained depth and drained depth obtained with fractal radiation and variable drainable porosity, using the head scheme.

Once the drains were closed, the water head was removed from the soil surface; the surface of the module was then covered with a plastic in order to avoid evaporation. Finally, the drains were opened to measure the drained water volume; it is noteworthy that the initial condition was equivalent to $h(x,0) = H_s$ and the recharge was null $R_w = 0$ during the drainage phase.

Soil porosity (ϕ) is calculated with the formula $\phi = 1 - \rho_t/\rho_s$, where ρ_t is the bulk density and ρ_s is the particles density; with the bulk density is determined from the weight and volume of the soil of drainage module $\rho_t = 1.14\ g/cm^3$ and the particles density $\rho_s = 2.65\ g/cm^3$, $\phi = 0.5695\ cm^3/cm^3$ is obtained. The value of $K_s = 1.15\ cm/h$ saturated hydraulic conductivity was estimated from a constant head test (Chávez, 2010). The soil fractal dimension obtained with Equation (18) is $s = 0.7026$. To minimize the root mean square error (RMSE) between the calculated drained depth with Boussinesq equation and the experimental drained depth, the conductance parameter was calibrated. Figure 8 presents the experimental drained depth compared to the calculated drained depth in function of time corresponding to fractal radiation condition and variable drainable porosity. Comparison showed that the drained depth experimental evolution with the drained depth obtained from the numerical solution is the same. The final value of conductance parameter is $\kappa = 0.0616$ with $\mathrm{RMSE} = 0.2195$ cm. These results indicate that water dynamics in a subterranean drainage system can be studied with the finite difference solution of the Boussinesq equation subject to fractal radiation in the drains and variable drainable porosity.

CONCLUSIONS

The non dimensional Boussinesq equation of the agricultural drainage systems has been solved using the finite differences method based on a mass local balance. Two discretization schemes of the time derivate were found: the mixed scheme and the head scheme. Both formulations are the same when the drainable porosity is independent of the head. Both schemes were validated with one analytical solution developed for lineal condition. The head water profile and the drained depth obtained with the analytical solution and calculated with the numerical solution, are the same for all the times tested, given an error criterion. The application of two schemes with the non lineal conditions is restricted. The absence of fluctuations for both in time and space to head and water depth, permit us to recommend the proposed numerical schemes for the non dimensional Boussinesq equation in the study of the water flow in the subterranean drainage systems.

In particular, the numerical solution development may be used to obtain the soil hydrodynamics characteristics through the inverse modelation; in other words, from the experimental data the parameter of system can be obtained. Furthermore, the numerical solution proposed in this investigation may be applied to the design of the subterranean drainage systems because the hypothesis used in the classical equation have been eliminated.

REFERENCES

Bear J (1972). Dynamics of Fluids in Porous Media. Dover Publications, Inc., New York, p. 764 .

Boussinesq J (1904). Recherches the´oriques sur l'e´coulement des nappes d'eau infiltre´es dans le sol et sur le de´bit des sources. J. Math. Pure. Appl. 5me. Ser., 10: 5-78.

Bouwer H (1964). Rapid field measurement of air entry value and hydraulic conductivity of soil as significant parameters in flow system analysis. Water Resour. Res., 36: 411-424.

Burdine NT (1953). Relative permeability calculation from size distribution data. Pet. Trans. AIME, 198: 71-78.

Chávez C (2010). Solución numérica de las ecuaciones de transferencia de agua y solutos en riego y drenaje (In Spanish). Dr. in Eng. Thesis. Universidad Autónoma de Querétaro. México, p. 200.

Dumm L (1954). Drain spacing formula. Agric. Eng., 35: 726-730.

Fragoza F, Fuentes C, Zavala M, Zataráin F, Saucedo H, Mejía E (2003). Subsurface agricultural drainage system with variable storage capacity (in spanish). Hydraul. Eng. México, 18(3): 81-93.

Fuentes C, Brambila F, Vauclin M, Parlange J-Y, Haverkamp R (2001). Fractal modeling of hydraulic conductivity in non saturated soil (in spanish). Hydraul. Eng. México, 16(2): 119-137.

Fuentes C, Haverkamp R, Parlange J-Y (1992). Parameter constraints on closed-form soil-water relationships. J. Hydrol., 134: 117-142.

Fuentes C, Namuche R, Rendón L, Patrón R, Palacios O, Brambila F, González A (1997). Solution of the Boussinesq equation of the transitional arrangements in agricultural drainage under conditions of radiation: The case of Valle del Carrizo, Sinaloa (in spanish). Hermosillo, Mexico, VII National Congress of Irrigation, 3: 141-145.

Fuentes C, Zavala M, Saucedo H (2009). Relationship between the Storage Coefficient and the Soil-Water Retention Curve in Subsurface Agricultural Drainage Systems: Water Table Drawdown. J. Irrigation and Drainage Eng., 135(3): 279-285.

Fujita H (1952). The exact pattern of a concentration-dependent difussion in a semi-infinite medium, part II. Textil Res. J., 22: 823-827.

Gardner WR (1958). Some steady-state solutions of the unsaturated moisture flow equation with application to evaporation from a water table. Soil Sci., 85: 228-232.

Hall FR, Moench AF (1972). Application of the convolution equation to stream-aquifer relationships. Water Resour. Res., 8 (2): 487-493.

Hogarth WL, Parlange J-Y, Parlange MB, Lockington D (1999). Approximate analytical solution of the Boussinesq equation with numerical validation. Water Resour. Res., 35(10): 3193-3194.

Hooghoudt S (1940). Contribute to the knowledge of any physical quantities of soil (in Dutch). Deel 7. Versl. Landb. Onderzoek, 46(7): 515-707.

Parlange J-Y, Braddock RD, Lisley I, Smith RE (1982). Three parameter infiltration equation. Soil Sci., 11: 170-174.

Richards LA (1931). Capillary conduction of liquids trough porous mediums. Physics, 1: 313-333.

Russell TF, Wheeler MF (1983). Finite element and finite difference methods for continuous flows in porous media. In The Mathematics of Reservoir Simulation. Ewing, R.E. (Ed.). Philadelphia, SIAM Publication, pp. 35-106.

Saucedo H, Pacheco P, Fuentes C, Zavala M (2003). Effect of the water table position on the evolution of the advance curve in border irrigation (in spanish). Hydraul. Eng. México, 18(4): 119-126.

Shukla KN, Chauhan HS, Srivastava VK (1999). Transient drainage to partially penetrating drains in sloping aquifers. J. Irrigation Drainage, 125 (5): 246-253.

Spanoudaki K, Nanou-Giannarou A, Paschalinos Y, Memos CD, Stamou (2010). Analytical solutions to the stream aquifer interaction problem: a critical review. Glob. NEST J., 12(2): 126-139.

Upadhyaya A, Chauhan HS (2001). Falling water tables in horizontal/sloping aquifer. J. Irrigation Drainage Eng., 127(6): 378-384.

Van-de-Giesen N, Steenhuis TS, Parlange J-Y (2005). Short and long time behavior of aquifer drainage after slow and sudden recharge according to the linearized Laplace equation. Adv. Water Resour., 28: 1122-1132.

Van-Genuchten M (1980). A closed-form equation for predicting the hydraulic conductivity of the unsaturated soils. Soil Sci. Soc. Am. J., 44: 892-898.

Zataráin F, Fuentes C, Palacios VOL, Mercado E, Brambila F, Villanueva N (1998). Modeling of solute and water transport in the soil. Agrociencia, 32(4): 373-383.

Zavala M, Fuentes C, Saucedo H (2007). Nonlinear radiation in the Boussinesq equation of agricultural drainage. J. Hydrol., 332(3): 374-380.